# On a Constructive Approximation of the Efficient Outcomes in Bicriterion Vector Optimization

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**Abstract.** For bicriterion quasiconvex optimization problems, we present a constructive procedure for an approximation of the efficient outcomes. Performing this procedure we can estimate the accuracy of the approximation. Conversely, if we prescribe an accuracy for the approximation, we can calculate the number of points which have to be computed by a certain scalarization method to remain under the given accuracy. Finally, we give a numerical example.

**Key words.** Bicriterion vector optimization, approximation of the efficient outcomes, interactive algorithm.

## 1. Motivation and Formulation of the Problem

Let two objective functions $f_k: \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2$, and a non-empty set $Z \subseteq \mathbb{R}^n$ be given. Then the vector optimization problem for the vector-valued objective function $f: \mathbb{R}^n \to \mathbb{R}^2$ with $f(x) := (f_1(x), f_2(x))$ for $x \in \mathbb{R}^n$ and the feasible set $Z$ is the problem of finding all efficient points in $Z$. Recall that a feasible point $v \in Z$ is called *efficient* iff

$$\underset{x \in Z}{\forall}\; f(x) \leqslant f(v) \Rightarrow f(x) = f(v)\,,$$

where the inequality and the equality are defined componentwise. Let $E$ denote the set of all efficient points in $Z$.

In general, $E$ is a set with infinitely many points and is too complicated to compute explicitly. Also an approximation of $E$ seems to be too unwieldly for the decision maker since it might be $n \geqslant 3$. Thus, our aim is to formulate a constructive procedure to compute an approximation of the efficient outcomes $f(E)$. In fact, by computing $E_m := \{v^1, \ldots, v^m\}$ by (17)–(19), we obtain

$$\sup_{v \in E} \inf_{i \in \{1,\ldots,m\}} \|f(v) - f(v^i)\|_2 \leqslant C\Delta_m\,, \tag{1}$$

where $\Delta_m \to 0$ as $m \to \infty$ and $C$ is constant only depending on $f_1$, $f_2$, and $Z$ (compare Corollary 4.2). Moreover, for a prescribed accuracy $\varepsilon$ for the approximation we can determine the number $m(\varepsilon) \in \mathbb{N}$ of points which are needed to achieve

$$\sup_{v \in E} \inf_{i \in \{1,\ldots,m(\varepsilon)\}} \|f(v) - f(v^i)\|_2 \leqslant \varepsilon \tag{2}$$

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(compare Corollary 4.3).

After computing a set $E_m$ with (1), (2) the approximation $f(E_m)$ of the efficient outcomes can be presented to the decision maker to select his solution(s) based on further informations. Thus, this enables us to formulate an interactive algorithm where the computer intensive part, namely the construction of $E_m$, is carried out first. Finally, the actual interactive part, where the decision maker communicates with the computer, can be realized very quickly.

To reach our goal we use the scalarization method (5). Many other authors also use this method (or a variant of it), see for example Aksoy [1], Bernau [4], Brosowski [5], Choo and Atkins [6], Helbig [10], Jahn [15], Jahn and Merkel [16], Shin *et al.* [19], and Steuer and Choo [21]). As in [7], [10], [15], [16], and [21], our purpose is to present a suitable sample of efficient points to the decision maker. In none of these papers, error estimates as given in (1) and (2) are given for these samples.

In our opinion, the scalarization method (5) has some advantages in contrast to some variants of it, for example the method use in [1], [3], [6], [20], [21], henceforeward called 'constraint method'. Firstly, a point in the feasible set $Z$ is also feasible for (5), but is in general not feasible for the constraint method. Secondly, by using the quantities $q_1 > 0$, $q_2 > 0$ in (5) (which can be interpreted as weights for the objective functions $f_1$, $f_2$) we are more flexible than in the constraint method. In fact, when rewriting (5) as a program in $\mathbb{R}^{n+1}$ with a linear objective function and with the functions $f_1$, $f_2$ in the constraints, (5) is exactly the constraint method by putting $q_1 = 1$, $q_2 = 0$, $p_1 = 0$. Thus, only the first objective function is positively weighted. Thirdly, by using (5) we are able to establish the error estimates (1), (2) for our procedure. One crucial result for this is the construction of the Lipschitz constant in Theorem 2.4, which would be infinity for the constraint method.

To attack our problem, we have to assume that

$$\left.\begin{array}{ll}\text{(a)} & f(E) \text{ is connected},\\ \text{(b)} & a_1 := \inf\limits_{x\in Z} f_1(x) \text{ and } b_2 := \inf\limits_{x \in Z} f_2(x) \text{ both exist and were attained, and}\\ \text{(c)} & f(Z) + \mathbb{R}^2_+ \text{ is closed}.\end{array}\right\} \tag{3}$$

Assumption (b) says that the objective functions are bounded from below on the feasible set. Moreover, together with (c), assumption (b) implies that the efficient point set is non-empty. Assumption (c) is fulfilled, if, for instance, $f$ is continuous and $Z$ is compact.

Of course, the topological assumption (a) is hard to verify for a concrete problem. Classes of vector optimization problems satisfying (a) are discussed in Choo and Atkins [7], Luc [14], or Helbig [9]. In particular, Schaible ([18] Theorem 2) establishes the following result:

THEOREM 1.1. *Let $Z$ be convex, compact and let $f_1$, $f_2$ be strictly quasiconvex on $Z$. Then $E$ and consequently $f(E)$ are connected.*

For our investigations, we need the following lemma. We omit the proof of it and also the proofs of the technical lemmata in Section 2. These results and their proofs are all geometrical motivated. In the working paper [11], they are drawn out in detail.

LEMMA 1.2. (*a*) *The numbers*

$$a_2 := \inf_{x \in P_1} f_2(x) \quad \text{and} \quad b_1 := \inf_{x \in P_2} f_1(x)$$

*were attained, where $P_k := \{v \in Z \mid f_k(v) = \min_{x \in Z} f_k(x)\}$ for $k = 1, 2$.*

*In the sequel, let $a := (a_1, a_2)$ and $b := (b_1, b_2)$, where $a_1$, $b_2$ are as required in assumption (3).*

(*b*) *For each $c = (c_1, c_2) \in \mathbb{R}^2$ with $c_1 \in [a_1, b_1]$ and $c_2 \in [b_2, a_2]$ and each $q = (q_1, a_2)$ with $q_1 > 0$, $q_2 > 0$, there exists unique $(\rho, t) \in [0, 1] \times \mathbb{R}$ such that*

$$c + tq = \rho a + (1 - \rho)b \tag{4}$$

*provided that $a \neq b$. In case of $a = b$, then $t = 0$ solves (4) uniquely for any $\rho \in [0, 1]$.* □

In the sequel, let $co(a, b)$ denote the convex hull of the points $a$ and $b$.

## 2. On a Scalarization Method

To solve the vector optimization problem, we introduce a scalarization method, which is due to Gemibicki [8] and, in its most general setting, to Pascoletti and Serafini [17]:

Let $q = (q_1, q_2)$ with $q_1 > 0$ and $q_2 > 0$ be fixed. For each $p \in co(a, b)$ consider the scalar optimization problem

$$\underline{OP(p)}: \min_{x \in Z} \max\left\{\frac{f_1(x) - p_1}{q_1}, \frac{f_2(x) - p_2}{q_2}\right\}. \tag{5}$$

Let $M(p)$ denote the minimal value and $P(p)$ the optimal set of the problem $OP(p)$, $p \in co(a, b)$. The relationships between solutions of $OP(p)$ and efficient points are given in ([17] Theorem 1.1) and in ([13] Theorem 1.1). In particular, we have

LEMMA 2.1. *If $p \in co(a, b)$ such that $f(P(p))$ is a singleton, then $P(p) \subseteq E$.* □

Under our assumptions (3)(b), (c), each scalar problem $MP(p)$ has a solution, and there are estimates for the minimal value $M(p)$ and for the images of the optimal points.

LEMMA 2.2. *Let assumptions (3)(b) and (c) be fulfilled. Then for each $p \in co(a, b)$*

$$(a)\ M(p) \leqslant \min\left\{\frac{b_1 - p_1}{q_1}, \frac{a_2 - p_2}{q_2}\right\};$$

$$(b)\ P(p) \neq \emptyset;$$

$$(c)\ f(v) \leqslant \begin{pmatrix} b_1 \\ a_2 \end{pmatrix} \text{ for each } v \in P(p). \qquad \square$$

In the next lemma, we establish that there are at most two efficient points in $P(p)$, $p \in co(a, b)$, and that they can be computed by a further minimization process.

LEMMA 2.3. *Let assumptions (3)(b) and (c) be fulfilled and let $p \in co(a, b)$. Define $v$ and $u$ in $Z$ as follows*:

$$\left.\begin{array}{l} v \in \arg\min\{f_2(x) \in \mathbb{R} \mid x \in P(p),\ f_1(x) = p_1 + M(p)q_1\} \\ u \in \arg\min\{f_1(x) \in \mathbb{R} \mid x \in P(p),\ f_2(x) = p_2 + M(p)q_2\}\,. \end{array}\right\} \tag{6}$$

*Then $v$ or $u$ or both are efficient and for each $w \in P(p) \cap E$ it follows $f(w) = f(v)$ or $f(w) = f(u)$.* $\square$

Next we establish the Lipschitz continuity of the minimal value mapping

$$M : co(a, b) \to \mathbb{R}, \quad p \mapsto M(p),$$

which is defined for each $p \in co(a, b)$ by Lemma 2.2(b).

THEOREM 2.4. *The mapping $M$ is Lipschitz continuous with the global Lipschitz constant*

$$L := \max\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\}.$$

*Especially, we have $|M(p) - M(p')| \leqslant \max_{k=1,2} |p_k - p'_k| q_k \leqslant L\|p - p'\|_2$ for each $p$, $p'$ in $co(a, b)$.* $\square$

This theorem in its most general setting can be found in Helbig ([12], Theorem 3.7).

## 3. Parametric Representation of the Efficient Point Set

In this section we show that the set $co(a, b)$ is a parametric representation of the efficient outcomes, where $a$ and $b$ are defined as in Lemma 1.2. Especially, $co(a, b)$ is homeomorphic to $f(E)$. Our approach extends other parametric representations of $E$, such as Benson ([3] Theorem 2.1) for convex problems or Choo and Atkins ([7] Lemma 4.1) for linear fractional problems. Schaible ([18] Lemma 2) generalizes the mentioned results in [3] and [7] to the case of $f_1$ being continuous and $f_2$ being strictly quasiconvex. But in this case there is no homeomorphism between the parameter set and $f(E)$. Related results can be found in Soland ([20] Theorem 1), Bacopoulos and Singer ([2] Theorem 2.1) and Aksoy ([1] Theorem 3).

THEOREM 3.1. *Let assumptions (3)(b) and (c) be fulfilled. For each $v \in E$ there exists an unique $p \in co(a, b)$ such that*

$$f(v) = p + M(p)q . \tag{7}$$

*Especially, $E \subseteq \bigcup_{p \in co(a,b)} P(p)$.*

*Proof.* By assumption, $E \neq \emptyset$. Let $v \in E$. Then $f_1(v) \in [a_1, b_1]$ and $f_2(v) \in [b_2, a_2]$. In fact, otherwise, it follows that $f(x) = a \leqslant f(v)$ (resp. $f(x) = b \leqslant f(v)$) for some $x \in Z$ with $f(x) \neq f(v)$. This would contradict the efficiency of $v$.

By Lemma 1.2(b), there exist unique numbers $t_0 \in \mathbb{R}$ and $p \in co(a, b)$ such that

$$f(v) - t_0 q = p . \tag{8}$$

We claim that

$$v \in P(p) \quad \text{and} \quad M(p) = t_0 . \tag{9}$$

By (8), it follows $M(p) \leqslant t_0$. Let $u \in P(p)$, which exists by Lemma 2.2(b). This implies that

$$f(u) \leqslant p + M(p)q \leqslant p + t_0 q = f(v) .$$

By efficiency of $v$, this implies $f(u) = f(v)$ and consequently also (9).

Since $v$ is arbitrary, (9) implies the assertion. □

For the converse of the inclusion in the last theorem we need assumption (3)(a).

THEOREM 3.2. *Let assumptions (3)(b) and (c) be fulfilled. Then the set $f(E)$ is connected, i.e. also (3)(a) holds, iff the mapping*

$$A: co(a, b) \to 2^{f(E)} , \quad p \mapsto f(P(p))$$

*is a point-to-point-mapping with* $A(p)=p+M(p)q$, *bijective, and continuous. Especially,* $E=\bigcup_{p\in co(a,b)} P(p)$ .

*Proof.* "$\Rightarrow$:" By the Lipschitz continuity of the mapping $M$ (compare Theorem 2.4) and by Theorem 3.1, it suffices to show that $A$ is a point-to-point-mapping, has the desired presentation, and is injective. To establish the former assertion, by Lemma 2.1, we have to show that $f(P(p))$ is a singleton for each $p\in co(a,b)$. Assume to the contrary that $f(P(p))$ contains more than one point for some $p\in co(a,b)$.

Let $v$, $u$ in $P(p)$ be constructed by (6) of Lemma 2.3. Without loss of generality let $v\in E$ and $f_2(v)<f_2(u)$. Moreover, let $\delta\in\mathbb{R}$ *be such that*

$$f_2(v)<\delta<f_2(u)=p_2+M(p)q_2\,. \tag{10}$$

Define a partition of $f(E)$ by

$$E_1:=\{y\in f(E)|y_2\leq\delta\}\quad\text{and}\quad E_2:=\{y\in f(E)\,|\,y_2>\delta\}\,.$$

Obviously, we have $E_1\cap E_2=\emptyset$ and $f(E)=E_1\cup E_2$. By Lemma 1.2(a), the points $a$ and $b$ are in $f(Z)$. Moreover, since $b_2\leq f_2(v)<\delta$ and, by Lemma 2.2(c), $\delta<f_2(u)\leq a_2$, we have $b\in E_1$ and $a\in E_2$, i.e. $E_1\neq\emptyset$ and $E_2\neq\emptyset$.

To establish the closedness of $E_1$ in $f(E)$, we claim that

$$y_2\leq f_2(v)\quad\text{for each}\quad y\in E_1\,. \tag{11}$$

In fact, if $\delta>f_2(\omega)>f_2(v)$ for some $f(w)\in E_1$, then, by efficiency of $w$, $f_1(w)<f_1(v)=p_1+M(p)q_1$. Since also $f_2(w)<\delta\leq p_2+M(p)q_2$ (compare (10)), we must have

$$\max_{k=1,2}\frac{f_k(w)-p_k}{q_k}<M(p)$$

in contradiction to the minimality of $M(p)$. Hence, (11) holds.

Now let $(f(w^k))\subseteq E_1$ be a sequence converging to some $f(w)\in f(E)$. By (11), $f_2(w)=\lim_{k\to\infty}f_2(w^k)\leq f_2(v)<\delta$, i.e., $f(w)\in E_1$. Therefore, $E_1$ is closed.

To establish the closedness of $E_2$ in $f(E)$, we claim that

$$y_2\geq f_2(u)\quad\text{for each}\quad y\in E_2\,.$$

Assume to the contrary that

$$\delta<f_2(w)<f_2(u)=p_2+M(p)q_2\quad\text{for some}\quad f(w)\in E_2\,. \tag{13}$$

If $f_1(w)\geq p_1+M(p)q_1$, then, by (13), (10) and (6), it follows $f(v)\leq f(w)$ and $f(v)\neq f(w)$ in contradiction to $w\in E$. Hence, $f_1(w)<p_1+M(p)q_1$. This together with (13) implies that

$$\max_{k=1,2} \frac{f_k(w) - p_k}{q_k} < M(p) .$$

Because of this contradiction (12) holds. The closedness of $E_2$ is now proved by the same arguments as above.

Thus, $f(E)$ would be not connected in contradiction to the assumption. Hence, $f(P(p))$ is a singleton and consequently $P(p) \subseteq E$.

By what we just proved, $A(p) = f(v)$ for each $v \in P(p)$. By construction (6) of Lemma 2.3, we obtain $A(p) = f(v) = p + M(p)q$.

To show the injectivity of $A$, let $p$, $p'$ be in $co(a, b)$ with $A(p) = A(p')$. There exist $w$, $w'$ in $E$ such that

$$p + M(p)q = f(w) = A(p) = A(p') = f(w') = p' + M(p')q .$$

Without loss of generality let $M(p) \leqslant M(p')$. Then

$$p - p' = (M(p') - M(p))q \geqslant \Theta . \tag{14}$$

Since $p$, $p'$ are in $co(a, b)$, we must have $p_k \leqslant p'_k$ and $p'_j \leqslant p_j$ for $k$, $j$ in $\{1, 2\}$ with $k \neq j$. Then (14) implies $p = p'$.

"$\Leftarrow$:" By continuity of $A$ and connectedness of $co(a, b)$, $f(E)$ is connected. □

By Theorems 3.1 and 3.2, we obtain the following corollaries.

COROLLARY 3.3. *Let assumption (3) be fulfilled. Then for each $v \in E$ there exists an unique $p \in co(a, b)$ such that*

$$f(v) = p + M(p)q . \tag{15}$$

*Conversely, for each $p \in co(a, b)$ there exists an unique $f(v) \in f(E)$ such that (15) holds.* □

COROLLARY 3.4. *Let assumption (3) be fulfilled. Then $f(E)$ is homeomorphic to $co(a, b)$.*

*Proof.* By Theorem 3.2, $A$ is continuous and bijective. The mapping $A^{-1}$ is also continuous since (8) may be rewritten to

$$\rho(b - a) + tq = f(v) - a$$

and the solution of this system depends continuously on the right hand side. □

## 4. Approximation of the Efficient Outcomes

In this section, we construct a discrete subset $E_m$ of $E$ satisfying (1), resp. (2). In particular, we obtain estimates of the distance between efficient points via 'their' parameters and of the number of points needed for $E_m$. Further results in this context (also for non-convex problems) can be found in Helbig ([13] Section 4). For the remaining part of the paper we assume that the vector $q=(q_1, q_2)$ with $q_1>0$, $q_2>0$ is such that

$$\langle q, b-a\rangle = 0\,. \tag{16}$$

In particular, this means that $\langle q, p-p'\rangle = 0$ for each $p$, $p'$ in $co(a,b)$. The estimates in (a) in the next results are more elegant but are coarser than those in (b).

We should remark that the estimates below can also be drawn without assumption (16). In this case the arising constants are bigger than those in the results below (see [11] for details).

THEOREM 4.1. *Let assumptions (3) and (16) be fulfilled. For each $p$, $p'$ in $co(a,b)$ and each $v \in P(p)$, $v' \in P(p)$ the following estimates hold*:

$$(a)\ \ \|f(v)-f(v')\|_2 \leqslant \sqrt{1+L^2\|q\|_2^2}\,\|p-p'\|_2\,,$$

$$(b)\ \ \|f(v)-f(v')\|_2 \leqslant \sqrt{\|p-p'\|_2^2 + L^2\|q\|_2^2\left(\max_{k=1,2}\frac{|p_k-p_k'|}{q_k}\right)^2}\,,$$

*where $L$ is the Lipschitz constant of Theorem* 2.4.

*Proof.* By Corollary 3.3, $f(v)=p+M(p)q$ and $f(v')=p'+M(p')q$. By (16), we obtain

$$\begin{aligned}\|f(v)-f(v')\|_2^2 &= \|p-p'+(M(p)-M(p'))q\|_2^2\\ &= \|p-p'\|_2^2 + (M(p)-M(p'))^2\|q\|_2^2\\ &\quad + 2(M(p)-M(p'))\langle p-p', q\rangle\\ &= \|p-p'\|_2^2 + (M(p)-M(p'))^2\|q\|_2^2\,.\end{aligned}$$

Now, estimating the last term with the constants of Theorem 2.4 it follows that

$$\begin{aligned}\|f(v)-f(v')\|_2^2 &\leqslant \|p-p'\|_2^2 + L^2\|q\|_2^2\left(\max_{i=1,2}\frac{|p_i-p_i'|}{q_i}\right)^2\\ &\leqslant (1+L^2\|q\|_2^2)\|p-p'\|_2^2\,.\end{aligned}$$

Thus, the proof is finished. □

Because of this result it makes sense to discretize the parameter set $co(a,b)$. Hence, let

$$P_m := \{p^j \in co(a,b) \mid j=1,\ldots,m\} \tag{17}$$

be a discretization of $co(a, b)$ with $m \in \mathbb{N}$, $p^1 = a$, $p^m = b$, and $p_1^j < p_1^{j+1}$ for $j = 1, \ldots, m-1$. Furthermore, let

$$\Delta_m = \max_{j=1,\ldots,m-1} \|p^{j+1} - p^j\|_2 , \quad \text{and}$$

$$\Delta_{m,k} = \max_{j=1,\ldots,m-1} |p_k^{j+1} - p_k^j| \quad \text{for} \quad k = 1, 2 . \tag{18}$$

For each $j \in \{1, \ldots, m\}$ we choose $v^j \in P(p^j)$ and define

$$E_m := \{v^1, \ldots, v^m\} . \tag{19}$$

COROLLARY 4.2. *Let assumptions (3) and (16) be fulfilled, let for $m \in \mathbb{N}$ the set $P_m$, the numbers $\Delta_m$ and $\Delta_{m,k}$ $k = 1, 2$, and the set $E_m$ be defined as in (17), (18), and (19). Then for each $v \in E$ there exists $v^i \in E_m$ such that the following estimates hold*:

$$\left.\begin{aligned} &(a)\ \|f(v) - f(v^i)\|_2 \leq \frac{\sqrt{1 + L^2\|q\|_2^2}}{2} \Delta_m , \\ &(b)\ \|f(v) - f(v^i)\|_2 \leq \tfrac{1}{2}\sqrt{\Delta_m^2 + L^2\|q\|_2^2\left(\max_{k=1,2} \frac{\Delta_{m,k}}{q_k}\right)^2} , \end{aligned}\right\} \tag{20}$$

*where $L$ is the Lipschitz constant of Theorem 2.4.*

*Proof.* Let $v \in E$. By Theorem 3.2, $f(P(p)) = \{f(v)\}$ for $p := A^{-1}(f(v))$. Let $j \in \{1, \ldots, m-1\}$ be such that $p \in co(p^j, p^{j+1})$. Then $p = \rho p^j + (1-\rho)p^{j+1}$ for some $\rho \in [0, 1]$. If $\rho \geq \frac{1}{2}$ let $i = j$. Otherwise let $i = j + 1$. In the former case, we obtain

$$\|p - p^i\|_2 = (1-\rho)\|p^i - p^{i+1}\|_2 \leq \frac{1}{2} \Delta_m .$$

In the latter case, it follows $\|p - p^i\|_2 = \rho\|p^{i-1} - p^i\|_2 \leq \frac{1}{2}\Delta_m$. Analogously, we show that $|p_k - p_k^i| \leq \Delta_{m,k}/2$ for $k = 1, 2$.

Consequently, using the estimates of Theorem 4.1, the assertions are proven. □

Next we want to remain under a certain accuracy $\varepsilon > 0$ in the last results or to approximate $f(E)$ with a prescribed $\varepsilon > 0$. Note that the for equidistant points in $P_m$, i.e.,

$$P_m := \left\{ a + \frac{j-1}{m-1} \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \middle| j = 1, \ldots, m \right\} , \tag{21}$$

it follows that

$$\left.\begin{aligned} \Delta_m &= \frac{\|b - a\|_2}{m-1} = \|p^j - p^{j+1}\|_2 \\ \Delta_{m,k} &= \frac{|b_k - a_k|}{m-1} = |p_k^j - p_k^{j+1}| \end{aligned}\right\} \text{for each} \quad j = 1, \ldots, m-1 \quad \text{and} \quad k = 1, 2 . \tag{22}$$

By Corollary 4.2 and (22), it follows immediately

COROLLARY 4.3. *Let assumption (3) be fulfilled. For given* $\varepsilon > 0$ *let*

$$\left.\begin{aligned} &(a)\ m(\varepsilon) \geqslant \frac{\sqrt{(1 + L\|q\|_2}\,\|b-a\|_2}{2\varepsilon} + 1\,,\\ &(b)\ m(\varepsilon) \geqslant \frac{\sqrt{\|b-a\|_2^2 + L^2\|q\|_2^2\left(\max\limits_{k=1,2} \dfrac{|b_k - a_k|}{q_k}\right)^2}}{2\varepsilon} + 1\,, \end{aligned}\right\} \tag{23}$$

*where* $L$ *is the Lipschitz constant of Theorem 2.4. Define the set* $P_{m(\varepsilon)}$ *of equidistant points, the numbers* $\Delta_{m(\varepsilon)}$ *and* $\Delta_{m(\varepsilon),k}$, $k = 1, 2$, *and the set* $E_{m(\varepsilon)}$ *as in (21), (22) and (19). Then*

$$\underset{v \in E}{\forall}\ \underset{i \in \{1,\ldots,m(\varepsilon)\}}{\exists}\ \|f(v) - f(v^i)\|_2 \leqslant \varepsilon\,. \qquad \square$$

Summarizing Corollaries 4.2 and 4.3, our desired results (1) and (2) are achieved. In fact, the accuracy of the approximation can be estimated by one of the constants in (20) of Corollary 4.2 by taking $E_m$ as defined in (19). On the other hand, if the accuracy $\varepsilon > 0$ is prescribed, then let $m(\varepsilon) \in \mathbb{N}$ be as in Corollary 4.3 and let $E_{m(\varepsilon)}$ as in (19). Then, the set $f(E_{m(\varepsilon)})$ approximates the efficient outcomes $f(E)$ with accuracy $\varepsilon$ if $P_{m(\varepsilon)}$ is as in (21).

Since the approximation of the efficient outcomes is constructive, we formulate the following procedure for performing this construction. Assume that (3) holds.

ALGORITHM TO CONSTRUCT AN APPROXIMATION OF THE EFFICIENT OUTCOMES WITH PRESCRIBED ACCURACY

(*Step 0*) Choose an accuracy $\varepsilon > 0$.

(*Step 1*) Determine the numbers $a_1$, $a_2$, $b_1$, $b_2$ and let $a = (a_1, a_2)$, $b = (b_1, b_2)$.
If $a = b$ then goto (Step 3).
Put $c := (b_1 - a_1, b_2 - a_2)$ and choose numbers $q_1 > 0$, $q_2 > 0$ such that (16) holds.

(*Step 2*) Calculate the number $m(\varepsilon) \in \mathbb{N}$ by (23).
Let $P_{m(\varepsilon)} := \{a + (j-1)/(m(\varepsilon) - 1)\ c \mid j = 1, \ldots, m(\varepsilon)\}$.
For each $j \in \{1, \ldots, m(\varepsilon)\}$ compute a solution $v^j \in Z$ of $OP(p^j)$.
Put $E_{m(\varepsilon)} := \{v^1, \ldots, v^{m(\varepsilon)}\}$.
Goto (Step 4).

(*Step 3*) Let $v \in Z$ be a solution of $OP(a)$.
Put $E_1 := \{v\}$ and $m(\varepsilon) := 1$.

(*Step 4*) The set $E_{m(\varepsilon)}$ is the desired subset of $E$ with property (2). STOP.

REMARK: In practice, to implement the above algorithm one would rewrite the scalar problem $OP(p^j)$ as a program in $\mathbb{R}^{n+1}$

$\underline{OP'(p^j)}$: $\min t$

subject to $x \in Z,\ t \in \mathbb{R}$,

$$f_k(x) - tq_k \leq p_k^j\,,\ k = 1, 2\,.$$

Thus, if we consider a convex vector optimization problem, i.e., $Z$ is a convex set and $f_k$, $k = 1, 2$, are convex functions, then $OP(p^j)$ is a convex program with linear objective function.

Based on an approximation of $f(E)$, an interactive algorithm can be formulated to help the decision maker to select his solution(s).

INTERACTIVE ALGORITHM

(*Step 0*) Calculate the points $a$ and $b$, the number $m(\varepsilon)$, and the sets $P_{m(\varepsilon)}$, $E_{m(\varepsilon)}$ by the above algorithm.

(*Step 1*) The decision maker chooses a reference point $p \in co(a, b)$. Calculate $i \in \{1, \ldots, m(\varepsilon)\}$ such that

$$\|p - p^i\|_2 \leqslant \|p - p^j\|_2 \quad \text{for each} \quad j = 1, \ldots, m(\varepsilon)\,. \tag{24}$$

The point $v^i$ is presented to the decision maker. If he does not accept this solution, repeat (Step 2), otherwise STOP.

REMARK. The choice of the number $i \in \{1, \ldots, m(\varepsilon)\}$ to guarantee the maximal error given in (20) can be made as shown in (24). In fact (excluding the trivial cases $a = b$ or $p = p^i$ for some $i \in \{1, \ldots, m(\varepsilon)\}$), since $p_1^j < p_1^{j+1}$ for $j = 1, \ldots, m(\varepsilon) - 1$, (20) can be realized by first calculating $j \in \{1, \ldots, m(\varepsilon) - 1\}$ such that

$$p_1^j < p_1 < p_1^{j+1}$$

and then putting $i = j$ or $i = j + 1$ in dependence whether $\|p - p^j\|_2 < \|p - p^{j+1}\|_2$ or not.

## 5. Some Numerical Examples

EXAMPLE 5.1. Let $n = 2$, $f_1(x) = x_1$, $f_2(x) = x_2$, $q = (4, 5)$, and $Z = \{x \in \mathbb{R}^2 \mid x_1 \leqslant 5,\ x_2 \leqslant 4,\ h(x) \geqslant 0\}$, where

$$h(x) := \min\{x_1^2 + (x_2 - 3)^2 - 1,\ x_1 + x_2 - 4,\ (x_1 - 2)^2 + (x_2 - 1)^2 - 1,\ x_1 + 2x_2 - 5\}\,.$$

Therefore, $E = f(E) = \{x \in Z \mid h(x) = 0\}$ is connected and $f(Z) + \mathbb{R}_+^2 = f(E) + \mathbb{R}_+^2$

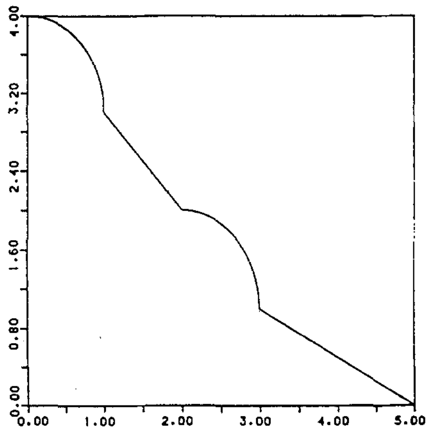


Fig. 1. The feasible set $Z$.

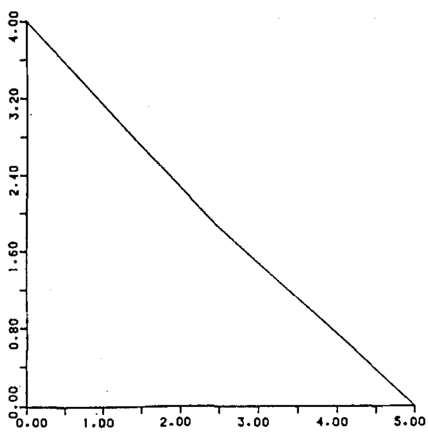


Fig. 2. The approximation for $m = 3$.

is closed. Further, we obtain $a = (0, 4)$, $b = (5, 0)$, and $L = 0.25$. Thus, when using equidistant parameters in $co(a, b)$, the maximal error is

| | $m = 3$ | $m = 11$ | $m = 51$ | $m = 101$ | $m = 1001$ |
|---|---|---|---|---|---|
| by (a) of (20) | 3.021 | 0.604 | 0.121 | 0.06 | 0.006 |
| by (b) of (20) | 2.562 | 0.512 | 0.102 | 0.05 | 0.005 |

By (a) or (b) of (23), we could calculate a lower bound for the number $m(\varepsilon)$ to remain under a given accuracy $\varepsilon > 0$. For $m = 3$, $m = 11$, and $m = 51$ we show the approximations $E_m = f(E_m)$, of $E = f(E)$ (see Figures 1–4).

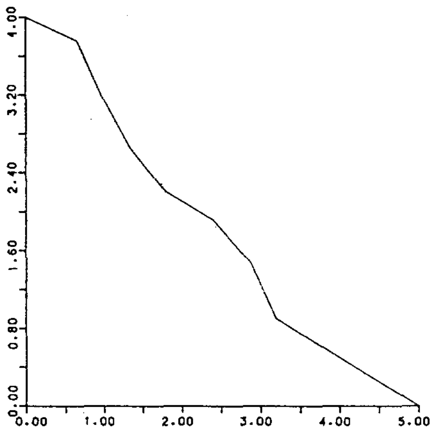


Fig. 3. The approximation for $m = 11$.

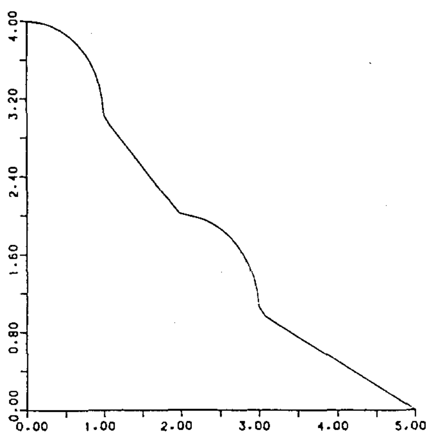


Fig. 4. The approximation for $m = 51$.

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